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1987 J. Phys. A: Math. Gen. 20 1081

(http://iopscience.iop.org/0305-4470/20/5/019)

J. Phys. A: Math. Gen. **20** (1987) 1081-1086. Printed in the UK

On the solution of a linear operator non-polynomial differential equation

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Received 7 May 1986

Abstract. A linear operator non-polynomial differential equation relativistically generalising the Airy equation and functions is considered. A fundamental system of two solutions of that equation representing generalised Airy functions of the first and second kind is obtained. Also, the general solution of another differential equation closely connected with the one considered is found.

In this paper we are concerned with the linear operator non-polynomial differential equation

$$(1-\alpha^2\,\mathrm{d}^2/\mathrm{d}x^2)^{1/2}y(x)+(\tfrac{1}{2}\alpha^2 x-1)y(x)=0 \qquad (1)$$

where α is a real positive constant, x is a real variable and the sign (i.e. the branch) of the square root operator† denotes its absolute eigenvalues. Because the operator $(1-\alpha^2\,\mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ is second order (Dimitrov 1981), equation (1) has two and not more than two linearly independent solutions (possessing Laplace integral representations), and in this sense it is an ordinary second-order differential equation‡. It may be seen that in the limit $\alpha \to 0^+$ the equation considered turns into the well known Airy equation (see, for instance, Abramowitz and Stegun 1970, Fedoryuk 1983)

$$\mathrm{d}^2y/\mathrm{d}x^2-xy=0. \qquad (2)$$

In this way, equation (1) is a generalisation of equation (2) and we shall call it a relativistic generalisation of the latter because of its connection with some problems of relativistic quantum mechanics.

Note that some other generalisations of the Airy equation and functions are done by Swanson and Headley (1967) and Kohno (1979) and by Sauter (1931) and Dimitrov (1982b), as well. By means of the extended form of the Airy equation

$$\mathrm{d}^n y/\mathrm{d}z^n - z^s y = 0$$

where n and s are positive integers and z is a complex variable, Swanson and Headley (1967) introduced the Airy functions of the first and second kind when $n=2$ and Kohno (1979) defined an Airy function of the first kind when n is an even number

† Note that such types of operators in the literature are known as pseudodifferential operators, too (see, for example, Treves 1982, Taylor 1981).

‡ Since the range of definition of a non-polynomial differential operator is the set of all infinitely differentiable functions, we assume that every one of the solutions of equation (1) should have a Laplace integral representation.

0305-4470/87/051081+06\$02.50

larger than 2. In the solving of a relativistic quantum problem with the help of the Dirac equation, Sauter (1931) obtained respective solutions essentially representing a generalisation of the Airy functions. Also in connection with a physical problem in a paper of ours (Dimitrov 1982b), we found the general solution of the linear operator non-polynomial differential equation of infinite order

$$[\cosh(a\mathrm{d}/\mathrm{d}x)+bx-c]y=0$$

where a, b and c are constants.

We shall seek the solutions of equation (1) when $x\in I=(-\infty,\infty)$ by means of the method proposed in our paper (Dimitrov 1982a). In accordance with the results of that paper (see equations (3) and (38)-(41) in it) the solutions of equation (1) are obtained in the form of the following Laplace integral representation:

$$y(x)=\int_C \mathrm{d}\tau\chi(\tau)\,\mathrm{e}^{x\tau} \tag{3}$$

where

$$\chi(\tau)=\text{constant}\times\exp\left[\frac{1}{\alpha^2}\left(\int_0^{\tau}\mathrm{d}\tau(1-\alpha^2\tau^2)^{1/2}-2\tau\right)\right]. \tag{4}$$

Here C is a path of integration in the complex τ plane $\mathbb{C}$ which is independent of x and assures the existence of the integral and is chosen so that the integrand in (3) should return to its initial value having passed along it. Moreover, because of the branch of the square root operator implied in equation (1), at any point τ of C the amplitude function $\chi(\tau)$ must satisfy the respective Laplace transform equation

$$\alpha^2\frac{\mathrm{d}\chi}{\mathrm{d}\tau}=2[(1-\alpha^2\tau^2)^{1/2}-1]\chi$$

for $\tau\in\mathbb{C}$ and $\tau\neq\pm1/\alpha$, ∞. Hence, the path of integration C must lie on one of the sheets of the Riemann surface of the integrand in (3) corresponding to the single-valued branch of the (operator generating) function

$$F(\tau)=(1-\alpha^2\tau^2)^{1/2} \tag{5}$$

defined by the condition $F(0)=1$.

From (3) and (4), after the computation of the integral in the exponent of the function (4), we obtain

$$y(x)=\text{constant}\int_C \mathrm{d}\tau\exp\left[\left(x-\frac{2}{\alpha^2}\right)\tau+\frac{1}{\alpha^3}[\alpha\tau(1-\alpha^2\tau^2)^{1/2}+\sin^{-1}(\alpha\tau)]\right]\qquad x\in I. \tag{6}$$

Now, it must be taken into account that the integrand in (6) is a multiple-valued function with branch points $\tau=-1/\alpha$, $\tau=1/\alpha$ and $\tau=\infty$. On every sheet of the Riemann surface of that function we shall regard the branch cuts as lying on the rays $\mathrm{Re}\,\tau=-1/\alpha$, $-\infty\leq\mathrm{Im}\,\tau\leq0$ and $\mathrm{Re}\,\tau=1/\alpha$, $0\leq\mathrm{Im}\,\tau\leq\infty$. Then the path of integration C can be chosen on any one of the sheets of the Riemann surface of the integrand in (6) which correspond to the principal branch of the square root function (5). Clearly, the choice must be done so that the path should start and end at the point at infinity going to it in the domains of the τ plane for which we have

$$\mathrm{Re}[\tau(1-\alpha^2\tau^2)^{1/2}]<0. \tag{7}$$

Choosing the path of integration C in a variety of ways (without leaving the sheet of the Riemann surface regarded) we shall obtain all (two) linearly independent solutions of equation (1) and possibly their linear combinations. Note that the solutions obtained in this way using different branches of the integrand in (6) with the principal branch of the square root function will differ from one another only by inessential constant factors.

Further, it is not difficult to see that inequality (7) is satisfied only in the two simply connected domains D_1 and D_2 defined by the conditions $0<\operatorname{Im}\tau\leqslant\infty$ when $-\infty\leqslant\operatorname{Re}\tau<-1/\alpha$ and $-\infty\leqslant\operatorname{Im}\tau\leqslant\infty$ when $-1/\alpha<\operatorname{Re}\tau<0$ for D_1 and $0<\operatorname{Im}\tau\leqslant\infty$ when $1/\alpha<\operatorname{Re}\tau\leqslant\infty$ for D_2. Therefore, the initial and final points of the path of integration C in (6), for example, can be chosen at infinity of the domain D_1 in the second and third quadrants of the τ plane, respectively, or contrariwise. Obviously, with an accuracy to the sign of the function obtained, all kinds of such paths are equivalent. Let us choose the path of integration C to start at infinity of the domain D_1 in the third quadrant and to end at infinity of the second quadrant and so that C should coincide with the imaginary axis of the complex τ plane. Then, putting $\operatorname{Im}\tau=t$, from (6) we obtain the following solution of equation (1):

$$y=\text{constant}\times y_1(\alpha;x)$$

for

$$y_1(\alpha;x)=\int_0^\infty \mathrm{d}t\cos\left[\left(x-\frac{2}{\alpha^2}\right)t+\frac{1}{\alpha^3}[\alpha t(1+\alpha^2t^2)^{1/2}+\sinh^{-1}(\alpha t)]\right] \qquad x\in I \tag{8}$$

where the functions $w^{1/2}$ and $\sinh^{-1} w$ are given by their principal branches.

Clearly, there are two other possibilities for the choice of the path of integration C in (6), namely, so that one of the ends of C should be at infinity of the domain D_2 and the other end should be at infinity of the domain D_1, either in the second or in the third quadrant of the complex τ plane. However, as may be seen, all paths C in (6) obtained by doing so are equivalent. Thus, from (6) it is possible to obtain only a second linearly independent solution of equation (1). Let us now choose the path of integration C to start at infinity of the domain D_1 in the third quadrant and to end at infinity of the domain D_2 of the complex τ plane and so that its parts should coincide with the negative imaginary axis, with the line segment $0\leqslant\operatorname{Re}\tau<1/\alpha$, $\operatorname{Im}\tau=0$, and, after surrounding the branch point $\tau=1/\alpha$, with the ray $1/\alpha<\operatorname{Re}\tau\leqslant\infty$, $\operatorname{Im}\tau=0$. Then, once again using everywhere t for the real-valued integration variable and taking into consideration that (8) is a solution of equation (1), from (6) we find the following second solution of that equation:

$$y=\text{constant}\times y_2(\alpha;x)$$

for

$$\begin{aligned}y_2(\alpha;x)=&\int_0^\infty \mathrm{d}t\sin\left[\left(x-\frac{2}{\alpha^2}\right)t+\frac{1}{\alpha^3}[\alpha t(1+\alpha^2t^2)^{1/2}+\sinh^{-1}(\alpha t)]\right]\\&+\int_0^{1/\alpha}\mathrm{d}t\exp\left[\left(x-\frac{2}{\alpha^2}\right)t+\frac{1}{\alpha^3}[\alpha t(1-\alpha^2t^2)^{1/2}+\sin^{-1}(\alpha t)]\right]\\&+\exp\left(\frac{\pi}{2\alpha^3}\right)\int_{1/\alpha}^\infty\mathrm{d}t\exp\left[\left(x-\frac{2}{\alpha^3}\right)t\right.\\&\left.+\frac{\mathrm{i}}{\alpha^3}[\alpha t(\alpha^2t^2-1)^{1/2}-\log(\alpha t+(\alpha^2t^2-1)^{1/2})]\right] \qquad x\in I.\end{aligned}\tag{9}$$

Here $i=(-1)^{1/2}$ and as in (8) each of the functions $w^{1/2}$, $\sinh^{-1} w$, $\sin^{-1} w$ and $\log w$ are given by their principal branches.

Therefore, the functions (8) and (9) form a fundamental system of solutions of equation (1) since the latter has no other linearly independent solutions. We notice that in the case of real x, the first solution (8) is real-valued but the second one (9) is a complex-valued function. Thus, the general solution of equation (1) may be written in the form

$$y(x)=C_1y_1(\alpha;x)+C_2y_2(\alpha;x) \tag{10}$$

where C_1 and C_2 are arbitrary (complex) constants of integration.

Furthermore, as is to be expected, in the limit $\alpha\to 0^+$ from (8) and (9) we obtain

$$\lim_{\alpha\to 0^+} y_1(\alpha;x)=\frac{\pi}{3^{1/3}}A_i(x) \qquad \lim_{\alpha\to 0^+} y_2(\alpha;x)=\frac{\pi}{3^{1/3}}B_i(x) \tag{11}$$

where $A_i(z)$ and $B_i(z)$ (generally, for a complex variable $z\in\mathbb{C}$) are the Airy functions of the first and second kind, respectively, given by the integral representations (Abramowitz and Stegun 1970)

$$A_i(z)=\frac{3^{1/3}}{\pi}\int_0^\infty \mathrm{d}t\cos(zt+\tfrac{1}{3}t^3) \qquad B_i(z)=\frac{3^{1/3}}{\pi}\int_0^\infty \mathrm{d}t[\sin(zt+\tfrac{1}{3}t^3)+\exp(zt-\tfrac{1}{3}t^3)].$$

For that reason, with an accuracy of constant factors, it is natural to introduce the symbols $A_i(\alpha;x)$ and $B_i(\alpha;x)$ for the functions $y_1(\alpha;x)$ and $y_2(\alpha;x)$ and to call them relativistic generalised Airy functions of the first and second kind, respectively. Then we should have

$$A_i(\alpha;x)=A_0y_1(\alpha;x) \qquad B_i(\alpha;x)=B_0y_2(\alpha;x)$$

and (12)

$$A_i(0;x)=A_i(x) \qquad B_i(0;x)=B_i(x)$$

where A_0 and B_0 are normalisation factors.

Now, when the solutions of equation (1) are known, it is easy to find also the solutions (having Laplace integral representations) of the differential equation

$$-(1-\alpha^2\,\mathrm{d}^2/\mathrm{d}x^2)^{1/2}y^{(-)}(x)+(\tfrac{1}{2}\alpha^2x-1)y^{(-)}(x)=0 \tag{13}$$

which is obtained from (1) by replacing the branch of the square root operator with the other one. Indeed, performing the substitutions of the independent variable x with ξ and the unknown function $y^{(-)}(x)$ with $\eta(\xi)$ by virtue of the equalities

$$x=\frac{4}{\alpha^2}-\xi \qquad y^{(-)}(x)=\eta(\xi) \tag{14}$$

we obtain from (13) a differential equation for the function $\eta(\xi)$ exactly of the same form as equation (1). Hence, taking into consideration the solutions of the latter, (8) and (9), and also the relations (12) and (14), we conclude that a fundamental system of solutions of equation (13) is given by the functions

$$y_1^{(-)}(x)=A_i(\alpha;\xi) \quad \text{and} \quad y_2^{(-)}(x)=B_i(\alpha;\xi) \quad \text{for } \xi=\frac{4}{\alpha^2}-x \quad x\in I. \tag{15}$$

Thereby, with the analogy of (10), the general solution of equation (13) may be expressed in the form

$$y^{(-)}(x) = C_1^{(-)} A_i(\alpha; 4/\alpha^2 - x) + C_2^{(-)} B_i(\alpha; 4/\alpha^2 - x) \tag{16}$$

where $C_1^{(-)}$ and $C_2^{(-)}$ are (complex) constants of integration. We should note that in contrast to (11), limits of the functions (15) and, consequently, of (16) do not exist when $\alpha \to 0^+$.

It is clear that the behaviour of the solutions of equation (1) $A_i(\alpha; x)$ and $B_i(\alpha; x)$ for $0 < \alpha$, obtained as functions of x, is quite similar to that of the Airy functions $A_i(x)$ and $B_i(x)$, respectively. Moreover, in accordance with (8), (9), (12), (15) and (16), the solutions of equations (1) and (13) are closely connected with each other. Thus, as must be expected, by changing the sign of $x - x_0$, where $x_0 = 2/\alpha^2$ is the turning point of equation (1), every solution $y(x)$ of that equation is converted into a corresponding solution $y^{(-)}(x)$ of equation (13) and conversely.

Now we note that all the considerations which we have done for finding the solutions of equations (1) and (13) are also valid in the case of a complex independent variable $z \in \mathbb{C}$ instead of the real one x. Obviously, the generalised Airy functions $A_i(\alpha; z)$ and $B_i(\alpha; z)$ obtained with exactness to constant factors from (8) and (9) and according to (12) by replacing the real variable x with a complex variable z are entire functions.

Finally, we may note also that it is not difficult to obtain the solutions of the more general differential equation of the form (1) as well where the real variable x is replaced by a complex variable $z \in \mathbb{C}$ and the parameter α is considered as an arbitrary given complex constant, different from zero. Then, as it is seen immediately, the form of the Laplace integral representation (6) remains valid for the solutions of the equation. Besides, without loss of generality, it is enough to suppose that $0 \leqslant |\arg \alpha| \leqslant \pi/2$. Furthermore, by the substitution of the integration variable τ with u, according to the equality $\tau = u \exp(-\mathrm{i} \arg \alpha)$, we reduce the integral representation (6) to the form in which the integrand is a multiple-valued function of u with branch points $u = 1/|\alpha|$, $u = 1/|\alpha|$ and $u = \infty$. The branch cuts for the integrand can then be selected in much the same manner as it was done above in the case of real positive α. However, now the path of integration C must be chosen on anyone of the sheets of the Riemann surface of the integrand corresponding to the principal branch of the square root function $(1 - |\alpha|^2 u^2)^{1/2}$ and so that the ends of C should go to the point at infinity in the domains of the complex u plane for which (instead of (7)) the inequality

$$|\alpha|^2 \operatorname{Re}\left(\frac{\tau}{\alpha^2}(1 - \alpha^2\tau^2)^{1/2}\right) = \operatorname{Re}[u(1 - |\alpha|^2 u^2)^{1/2} \exp(-3\mathrm{i} \arg \alpha)] < 0$$

is satisfied. In this way, by choosing all the possible non-equivalent paths of integration C we shall find a fundamental system of two (complex-valued) solutions of the equation. From that system of solutions we could also immediately obtain the general solution of the differential equation of the form (13), however, with a complex independent variable $z \in \mathbb{C}$ instead of x and with a complex parameter $\alpha \neq 0$ by replacing z with $4/\alpha^2 - z$. Clearly, all the solutions obtained in this way will be entire functions of z.

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